

GENERALIZED SPIN REPRESENTATIONS. PART 2: CARTAN–BOTT PERIODICITY FOR THE SPLIT REAL E_n SERIES

MAX HORN AND RALF KÖHL (NÉ GRAMLICH)

ABSTRACT. In this article we analyze the quotients of the maximal compact subalgebras of the split real Kac–Moody algebras of the E_n series resulting from the generalized spin representation introduced in [HKL13]. It turns out that these quotients satisfy a Cartan–Bott periodicity.

Our findings are also meaningful in the finite-dimensional cases of $A_2 \oplus A_1$, A_4 , D_5 , E_6 , E_7 , E_8 , where it turns out that the generalized spin representation is injective. Consequently the observed Cartan–Bott periodicity provides a structural explanation for the seemingly sporadic isomorphism types of the maximal compact Lie subalgebras of the split real Lie algebras of types E_6 , E_7 , E_8 .

1. INTRODUCTION

In this article we continue the investigation of the generalized spin representations introduced in the first part [HKL13]. We focus on the E_n series and use the original description of the generalized spin representation from [DKN06], [DBHP06], [HKL13] via Clifford algebras.

The E_n series is traditionally only defined for $n \in \{6, 7, 8\}$. However, using the Bourbaki style labeling shown in Figure 1, it naturally extends to arbitrary $n \in \mathbb{N}$. Using this description, one has $E_1 = A_1$, $E_2 = A_1 \oplus A_1$, $E_3 = A_2 \oplus A_1$, $E_4 = A_4$, $E_5 = D_5$ (see Figure 2).

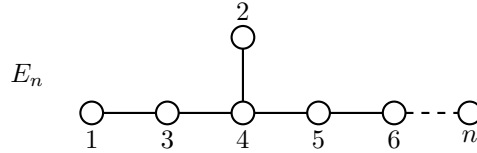


FIGURE 1. The Dynkin diagram of type E_n

An elementary combinatorial counting argument using binomial coefficients allows us to determine lower bounds for the \mathbb{R} -dimension of the images of the generalized spin representation. These images have to be compact, whence reductive by [HKL13, Theorem 4.11] and even semisimple, if the diagram be irreducible, thus providing an upper bound for the \mathbb{R} -dimension via the maximal compact Lie subalgebras of the Clifford algebras. As it turns out, the lower and the upper bounds coincide, providing the following Cartan–Bott periodicity.

Theorem A (Cartan–Bott periodicity of the E_n series). *Let $n \in \mathbb{N}$ with $n \geq 4$, let \mathfrak{k} be the maximal compact Lie subalgebra of the split real Kac–Moody Lie algebra of type E_n , let $C = C(\mathbb{R}^n, q)$ be the Clifford algebra with respect to the standard positive definite quadratic form q and let $\rho : \mathfrak{k} \rightarrow C$ be the standard generalized spin representation.*

Then $\text{im}(\rho)$ is isomorphic to

- (0) $\mathfrak{so}(2^{\frac{n}{2}}) \leq \mathbb{R} \otimes_{\mathbb{R}} M(2^{\frac{n}{2}}, \mathbb{R})$, if $n \equiv 0 \pmod{8}$,
- (1) $\mathfrak{so}(2^{\frac{n-1}{2}}) \oplus \mathfrak{so}(2^{\frac{n-1}{2}}) \leq (\mathbb{R} \oplus \mathbb{R}) \otimes_{\mathbb{R}} M(2^{\frac{n-1}{2}}, \mathbb{R})$, if $n \equiv 1 \pmod{8}$,
- (2) $\mathfrak{so}(2^{\frac{n}{2}}) \leq M(2, \mathbb{R}) \otimes_{\mathbb{R}} M(2^{\frac{n-2}{2}}, \mathbb{R})$, if $n \equiv 2 \pmod{8}$,
- (3) $\mathfrak{su}(2^{\frac{n-1}{2}}) \leq M(2, \mathbb{C}) \otimes_{\mathbb{R}} M(2^{\frac{n-3}{2}}, \mathbb{R})$, if $n \equiv 3 \pmod{8}$,
- (4) $\mathfrak{sp}(2^{\frac{n-2}{2}}) \leq M(2, \mathbb{H}) \otimes_{\mathbb{R}} M(2^{\frac{n-4}{2}}, \mathbb{R})$, if $n \equiv 4 \pmod{8}$,
- (5) $\mathfrak{sp}(2^{\frac{n-3}{2}}) \oplus \mathfrak{sp}(2^{\frac{n-3}{2}}) \leq (M(2, \mathbb{H}) \oplus M(2, \mathbb{H})) \otimes_{\mathbb{R}} M(2^{\frac{n-5}{2}}, \mathbb{R})$, if $n \equiv 5 \pmod{8}$,
- (6) $\mathfrak{sp}(2^{\frac{n-2}{2}}) \leq M(4, \mathbb{H}) \otimes_{\mathbb{R}} M(2^{\frac{n-6}{2}}, \mathbb{R})$, if $n \equiv 6 \pmod{8}$,
- (7) $\mathfrak{su}(2^{\frac{n-1}{2}}) \leq M(8, \mathbb{C}) \otimes_{\mathbb{R}} M(2^{\frac{n-7}{2}}, \mathbb{R})$, if $n \equiv 7 \pmod{8}$,

i.e., $\text{im}(\rho)$ is a semisimple maximal compact Lie subalgebra of C .

Along the way we arrive at a structural explanation for the isomorphism types of the maximal compact Lie subalgebras of the semisimple split real Lie algebras of types $E_3 = A_2 \oplus A_1$, $E_4 = A_4$, $E_5 = D_5$, E_6 , E_7 , E_8 .

Theorem B. *The maximal compact Lie subalgebras of the semisimple split real Lie algebras of types $A_2 \oplus A_1$, A_4 , D_5 , E_6 , E_7 , E_8 are isomorphic to $\mathfrak{u}(2)$, $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$, $\mathfrak{sp}(2) \oplus \mathfrak{sp}(2) \cong \mathfrak{so}(5) \oplus \mathfrak{so}(5)$, $\mathfrak{sp}(4)$, $\mathfrak{su}(8)$, $\mathfrak{so}(16)$, respectively.*

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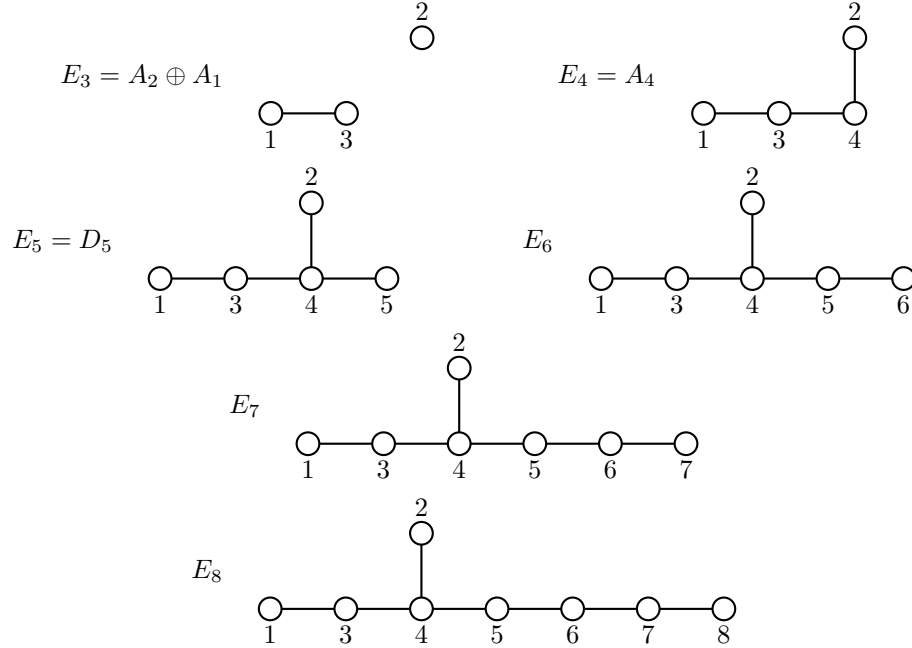


FIGURE 2. The Dynkin diagrams of types E_3 to E_8 .

2. CARTAN-BOTT PERIODICITY OF CLIFFORD ALGEBRAS

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of natural numbers, and let \mathbb{R} , \mathbb{C} , resp. \mathbb{H} denote the reals, complex numbers resp. quaternions. For $n \in \mathbb{N}$ and a division ring \mathbb{D} , denote by $M(n, \mathbb{D})$ the \mathbb{D} -algebra of $n \times n$ matrices over \mathbb{D} .

Let V be an \mathbb{R} -vector space and $q: V \rightarrow \mathbb{R}$ a quadratic form with associated bilinear form b . Then the **Clifford algebra** $C(V, q)$ is defined as $C(V, q) := T(V)/\langle vw + wv - b(v, w) \rangle$ where $T(V)$ is the tensor algebra of V ; cf. [KY05, Section 4.3], [LM89, Chapter 1, §1].

Let $V = \mathbb{R}^n$ with standard basis vectors v_i , let $q = x_1^2 + \dots + x_n^2$. Then in $C(V, q)$ we have $v_i^2 = 1$ and $v_i v_j = -v_j v_i$.

Proposition 2.1 (Cartan–Bott periodicity). *For $n \geq 2$, the Clifford algebra $C(\mathbb{R}^n, q)$ is isomorphic to the following algebra:*

- (0) $\mathbb{R} \otimes_{\mathbb{R}} M(2^{\frac{n}{2}}, \mathbb{R})$, if $n \equiv 0 \pmod{8}$,
- (1) $(\mathbb{R} \oplus \mathbb{R}) \otimes_{\mathbb{R}} M(2^{\frac{n-1}{2}}, \mathbb{R})$, if $n \equiv 1 \pmod{8}$,
- (2) $M(2, \mathbb{R}) \otimes_{\mathbb{R}} M(2^{\frac{n-2}{2}}, \mathbb{R})$, if $n \equiv 2 \pmod{8}$,
- (3) $M(2, \mathbb{C}) \otimes_{\mathbb{R}} M(2^{\frac{n-3}{2}}, \mathbb{R})$, if $n \equiv 3 \pmod{8}$,
- (4) $M(2, \mathbb{H}) \otimes_{\mathbb{R}} M(2^{\frac{n-4}{2}}, \mathbb{R})$, if $n \equiv 4 \pmod{8}$,
- (5) $(M(2, \mathbb{H}) \oplus M(2, \mathbb{H})) \otimes_{\mathbb{R}} M(2^{\frac{n-5}{2}}, \mathbb{R})$, if $n \equiv 5 \pmod{8}$,
- (6) $M(4, \mathbb{H}) \otimes_{\mathbb{R}} M(2^{\frac{n-6}{2}}, \mathbb{R})$, if $n \equiv 6 \pmod{8}$,
- (7) $M(8, \mathbb{C}) \otimes_{\mathbb{R}} M(2^{\frac{n-7}{2}}, \mathbb{R})$, if $n \equiv 7 \pmod{8}$.

Proof. See e.g. [KY05, Proposition 4.4.1 + Table 4.4.1]. □

Since $C(V, q)$ is an associative algebra, it becomes a Lie algebra by setting $[A, B] := AB - BA$. With this in mind, Proposition 2.1 implies the following:

Corollary 2.2. *For $n \geq 2$, the maximal semisimple compact Lie subalgebra of the Clifford algebra $C(\mathbb{R}^n, q)$ is isomorphic to the following Lie algebra:*

- (0) $\mathfrak{so}(2^{\frac{n}{2}})$, if $n \equiv 0 \pmod{8}$,
- (1) $\mathfrak{so}(2^{\frac{n-1}{2}}) \oplus \mathfrak{so}(2^{\frac{n-1}{2}})$, if $n \equiv 1 \pmod{8}$,
- (2) $\mathfrak{so}(2^{\frac{n}{2}})$, if $n \equiv 2 \pmod{8}$,
- (3) $\mathfrak{su}(2^{\frac{n-1}{2}})$, if $n \equiv 3 \pmod{8}$,
- (4) $\mathfrak{sp}(2^{\frac{n-2}{2}})$, if $n \equiv 4 \pmod{8}$,
- (5) $\mathfrak{sp}(2^{\frac{n-3}{2}}) \oplus \mathfrak{sp}(2^{\frac{n-3}{2}})$, if $n \equiv 5 \pmod{8}$,
- (6) $\mathfrak{sp}(2^{\frac{n-2}{2}})$, if $n \equiv 6 \pmod{8}$,
- (7) $\mathfrak{su}(2^{\frac{n-1}{2}})$, if $n \equiv 7 \pmod{8}$.

3. A LOWER BOUND ON THE DIMENSION OF A SUBALGEBRA

Definition 3.1. For $n \geq 3$ let \mathfrak{m} be the Lie subalgebra of $C(\mathbb{R}^n, q)$ generated by $v_1 v_2 v_3$ and by $v_i v_{i+1}$, $1 \leq i < n$.

Lemma 3.2. *Let $n \geq 3$. Then \mathfrak{m} contains all products of the form $v_{j_1} v_{j_2} \dots v_{j_k}$ for $2 \leq k \leq n$ and $k \equiv 2, 3 \pmod{4}$ with pairwise distinct $j_t \in \{1, \dots, n\}$, with the possible exception of $v_1 v_2 \dots v_n$. The exception can only happen if $n \equiv 3 \pmod{4}$.*

Proof. It is well-known that all products $v_{j_1} v_{j_2}$, $j_1 \neq j_2$, are contained in \mathfrak{m} : Indeed, $\Lambda^2 \mathbb{R}^n \cong \mathfrak{so}(n)$ (cf., e.g., [LM89, Proposition 6.1]) is generated as a Lie algebra by the $v_i v_{i+1}$, $1 \leq i < n$ (cf., e.g., [Ber89, Theorem 1.31], [HKL13, Theorem 2.1]).

Moreover, for pairwise distinct j_t , $1 \leq t \leq k+1$, one has

$$[v_{j_1} v_{j_2}, v_{j_2} v_{j_3} \cdots v_{k+1}] = 2v_{j_1} v_{j_3} \cdots v_{j_{k+1}}.$$

Since re-ordering of the factors simply yields scalar multiples, this shows inductively that, as long as $k+1 \leq n$, once an arbitrary factor of the form $v_{j_1} v_{j_2} \cdots v_{j_k}$ is contained in the Lie subalgebra, all factors of that form are contained in the Lie subalgebra. This statement is also true in the situation $k = n$, because in that case all factors of that form are scalar multiples of one another.

We finally prove the claim by induction over k . For $k = 2$ and $k = 3$, this is obvious. Suppose the claim holds for $k \equiv 3 \pmod{4}$, then the next value for k to consider is $k+3 \equiv 2 \pmod{4}$. By induction hypothesis $v_4 v_5 \cdots v_{k+3} \in \mathfrak{m}$ and

$$0 \neq [v_1 v_2 v_3, v_4 v_5 \cdots v_{k+3}] = 2v_1 v_2 v_3 v_4 \cdots v_{k+3}.$$

If on the other hand the claim holds for $k \equiv 2 \pmod{4}$, then the next value for k to consider is $k+1 \equiv 3 \pmod{4}$. If $k+2 \leq n$, then by induction hypothesis $v_3 v_4 \cdots v_{k+2} \in \mathfrak{m}$ and

$$0 \neq [v_1 v_2 v_3, v_3 v_4 \cdots v_{k+2}] = 2v_1 v_2 v_4 \cdots v_{k+2}.$$

That is, the presence of all elements of the form $v_{j_1} v_{j_2} v_{j_3}$ with pairwise distinct $j_t \in \{1, \dots, n\}$ inductively allows us to construct all elements of the form $v_{j_1} v_{j_2} \cdots v_{j_k}$ for $k \equiv 2, 3 \pmod{4}$ with pairwise distinct $j_t \in \{1, \dots, n\}$ for all $k \leq n$, with the possible exception of the situation $k = n \equiv 3 \pmod{4}$, as the element v_{k+2} does not exist in that case. \square

Remark 3.3. It will turn out later, as a consequence of the proof of Theorem A based on dimension arguments, that the above elements in fact generate \mathfrak{m} as an \mathbb{R} -vector space and that for $n \equiv 3 \pmod{4}$ the element $v_1 v_2 \cdots v_n$ indeed is not contained in \mathfrak{m} , unless of course $n = 3$.

Definition 3.4. For $k \in \{0, 1, 2, 3\}$, let

$$\delta_k : \mathbb{N} \rightarrow \mathbb{N} : n \mapsto \sum_{\substack{i=0, \\ i \equiv k \pmod{4}}}^n \binom{n}{i}.$$

Remark 3.5. Let $n \in \mathbb{N}$ and let M be a set of size n . Then the number of subsets of M of size $k \pmod{4}$ is precisely $\delta_k(n)$. Therefore

$$\delta_0(n) + \delta_1(n) + \delta_2(n) + \delta_3(n) = 2^n.$$

Consequence 3.6. Let $n \geq 3$. Then

$$\dim \mathfrak{m} \geq \begin{cases} \delta_2(n) + \delta_3(n) & \text{if } n \not\equiv 3 \pmod{4}, \\ \delta_2(n) + \delta_3(n) - 1 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

4. COMBINATORICS OF BINOMIAL COEFFICIENTS

We now turn the lower bound from Consequence 3.6 into a numerically explicit bound by deriving a closed formula in n for the functions δ_k .

Proposition 4.1. Let $n \in \mathbb{N}$ and $k \in \{0, 1, 2, 3\}$.

(0) If $n \equiv 0 \pmod{4}$, then

$$\delta_k(n) = \begin{cases} 2^{n-2} & \text{for } k \in \{1, 3\}, \\ 2^{n-2} + (-1)^{\frac{n}{4} + \frac{k}{2}} 2^{\frac{n}{2}-1} & \text{for } k \in \{0, 2\}. \end{cases}$$

(1) If $n \equiv 1 \pmod{4}$, then

$$\delta_k(n) = \begin{cases} 2^{n-2} + (-1)^{\frac{n-1}{4}} 2^{\frac{n-3}{2}} & \text{for } k \in \{0, 1\}, \\ 2^{n-2} - (-1)^{\frac{n-1}{4}} 2^{\frac{n-3}{2}} & \text{for } k \in \{2, 3\}. \end{cases}$$

(2) If $n \equiv 2 \pmod{4}$, then

$$\delta_k(n) = \begin{cases} 2^{n-2} & \text{for } k \in \{0, 2\}, \\ 2^{n-2} + (-1)^{\frac{n-2}{4} + \frac{k-1}{2}} 2^{\frac{n}{2}-1} & \text{for } k \in \{1, 3\}. \end{cases}$$

(3) If $n \equiv 3 \pmod{4}$, then

$$\delta_k(n) = \begin{cases} 2^{n-2} - (-1)^{\frac{n-3}{4}} 2^{\frac{n-3}{2}} & \text{for } k \in \{0, 3\}, \\ 2^{n-2} + (-1)^{\frac{n-3}{4}} 2^{\frac{n-3}{2}} & \text{for } k \in \{1, 2\}. \end{cases}$$

Proof. Note first that the claimed identities hold for $n \in \{1, 2\}$. The pairing $S \leftrightarrow S \triangle \{n\}$, where \triangle denotes symmetric difference, provides a bijection between the set of subsets of M of even order with the set of subsets of M of odd order. Combined with Remark 3.5 we conclude

$$(1) \quad \delta_0(n) + \delta_2(n) = \delta_1(n) + \delta_3(n) = 2^{n-1}.$$

Moreover, the pairing $S \leftrightarrow M \setminus S$ provides a bijection

- (i) between the set of subsets of M of order 1 (mod 4) and the set of subsets of M of order 3 (mod 4), if $n \equiv 0 \pmod{4}$,
- (ii) between the set of subsets of M of order 0 (mod 4) and the set of subsets of M of order 1 (mod 4) and between the set of subsets of M of order 2 (mod 4) and the set of subsets of M of order 3 (mod 4), if $n \equiv 1 \pmod{4}$,
- (iii) between the set of subsets of M of order 0 (mod 4) and the set of subsets of M of order 2 (mod 4), if $n \equiv 2 \pmod{4}$,
- (iv) between the set of subsets of M of order 0 (mod 4) and the set of subsets of M of order 3 (mod 4) and between the set of subsets of M of order 1 (mod 4) and the set of subsets of M of order 2 (mod 4), if $n \equiv 3 \pmod{4}$.

Hence

$$\begin{aligned} \delta_1(n) &= \delta_3(n) & \text{for } n \equiv 0 \pmod{4}, \\ \delta_0(n) &= \delta_1(n) \quad \text{and} \quad \delta_2(n) = \delta_3(n) & \text{for } n \equiv 1 \pmod{4}, \\ \delta_0(n) &= \delta_2(n) & \text{for } n \equiv 2 \pmod{4}, \\ \delta_0(n) &= \delta_3(n) \quad \text{and} \quad \delta_1(n) = \delta_2(n) & \text{for } n \equiv 3 \pmod{4}. \end{aligned}$$

Together with Equation 1, this already yields the claim for (a), case $k \in \{1, 3\}$ and for (c), case $k \in \{0, 2\}$.

We will now prove case $k = 0$ of (b), (d) by induction, which by the above observations implies all claims made in (b), (d). Let M be a set of order $n + 2$ and let $a, b \in M$ be distinct elements so that $M = M' \cup \{a, b\}$ for a set M' of cardinality n . A subset $S \subset M$ of cardinality 0 (mod 4) satisfies exactly one of the following:

- (i) $S \subset M'$ has cardinality 0 (mod 4),
- (ii) $S \setminus \{a\} \subset M'$ has cardinality 3 (mod 4),
- (iii) $S \setminus \{b\} \subset M'$ has cardinality 3 (mod 4),
- (iv) $S \setminus \{a, b\} \subset M'$ has cardinality 2 (mod 4).

Hence for $n \equiv 1 \pmod{4}$ resp. $n + 2 \equiv 3 \pmod{4}$ we have

$$\begin{aligned} \delta_0(n+2) &= \delta_0(n) + \delta_2(n) + 2\delta_3(n) = 2^{n-1} + 2\delta_3(n) \\ &= 2^{n-1} + 2 \left(2^{n-2} - (-1)^{\frac{n-1}{4}} 2^{\frac{n-3}{2}} \right) \\ &= 2^n - (-1)^{\frac{n-1}{4}} 2^{\frac{n-1}{2}} \\ &= 2^{(n+2)-2} - (-1)^{\frac{(n+2)-3}{4}} 2^{\frac{(n+2)-3}{2}}, \end{aligned}$$

and similarly for $n \equiv 3 \pmod{4}$ resp. $n + 2 \equiv 1 \pmod{4}$ we have

$$\begin{aligned}\delta_0(n+2) &= \delta_0(n) + \delta_2(n) + 2\delta_3(n) = 2^{n-1} + 2\delta_3(n) \\ &= 2^{n-1} + 2 \left(2^{n-2} - (-1)^{\frac{n-3}{4}} 2^{\frac{n-3}{2}} \right) \\ &= 2^n - (-1)^{\frac{n-3}{4}} 2^{\frac{n-1}{2}} \\ &= 2^{(n+2)-2} + (-1)^{\frac{(n+2)-1}{4}} 2^{\frac{(n+2)-3}{2}}.\end{aligned}$$

Next we prove case $k = 0$ of (a) using (c) as an induction hypothesis and afterwards case $k = 1$ of (c) using (a) as an induction hypothesis. By the above observations this implies all claims made in (a) and (c).

In order to establish case $k = 0$ of (a) we use the exact same combinatorial induction step as above and arrive again at

$$\begin{aligned}\delta_0(n+2) &= \delta_0(n) + \delta_2(n) + 2\delta_3(n) = 2^{n-1} + 2\delta_3(n) \\ &= 2^{n-1} + 2 \left(2^{n-2} + (-1)^{\frac{n-2}{4} + \frac{3-1}{2}} 2^{\frac{n}{2}-1} \right) \\ &= 2^n + (-1)^{\frac{n+2}{4}} 2^{\frac{n}{2}} \\ &= 2^{(n+2)-2} + (-1)^{\frac{n+2}{4} + \frac{0}{2}} 2^{\frac{n+2}{2}-1}\end{aligned}$$

as claimed.

In order to establish case $k = 1$ of (c) we use the same combinatorial induction step as above but need to observe that if $S \subset M$ is a subset of cardinality $1 \pmod{4}$, then $S \setminus \{a, b\}$ may have cardinality $1 \pmod{4}$, $3 \pmod{4}$ or, in two different ways, $0 \pmod{4}$. Therefore

$$\begin{aligned}\delta_1(n+2) &= 2\delta_0(n) + \delta_1(n) + \delta_3(n) = 2\delta_0(n) + 2^{n-1} \\ &= 2 \left(2^{n-2} + (-1)^{\frac{n}{4} + \frac{0}{2}} 2^{\frac{n}{2}-1} \right) + 2^{n-1} \\ &= 2^n + (-1)^{\frac{n}{4} + \frac{0}{2}} 2^{\frac{n}{2}} \\ &= 2^{(n+2)-2} + (-1)^{\frac{(n+2)-2}{4} + \frac{1-1}{2}} 2^{\frac{n+2}{2}-1}.\end{aligned}$$

□

Combining this with Consequence 3.6 yields the following:

Consequence 4.2. *Let $n \in \mathbb{N}$ and $n \geq 2$.*

(0) *If $n \equiv 0 \pmod{8}$, then*

$$\begin{aligned}\dim \mathfrak{m} &\geq \delta_2(n) + \delta_3(n) = 2^{n-2} - 2^{\frac{n}{2}-1} + 2^{n-2} = 2^{\frac{n-2}{2}} (2^{\frac{n}{2}} - 1) \\ &= \dim_{\mathbb{R}}(\mathfrak{so}(2^{\frac{n}{2}})).\end{aligned}$$

(1) *If $n \equiv 1 \pmod{8}$, then*

$$\begin{aligned}\dim \mathfrak{m} &\geq \delta_2(n) + \delta_3(n) = 2 \left(2^{n-2} - 2^{\frac{n-3}{2}} \right) = 2^{\frac{n-1}{2}} (2^{\frac{n-1}{2}} - 1) \\ &= \dim_{\mathbb{R}}(\mathfrak{so}(2^{\frac{n-1}{2}}) \oplus \mathfrak{so}(2^{\frac{n-1}{2}})).\end{aligned}$$

(2) *If $n \equiv 2 \pmod{8}$, then*

$$\begin{aligned}\dim \mathfrak{m} &\geq \delta_2(n) + \delta_3(n) = 2^{n-2} + 2^{n-2} - 2^{\frac{n}{2}-1} = 2^{\frac{n-2}{2}} (2^{\frac{n}{2}} - 1) \\ &= \dim_{\mathbb{R}}(\mathfrak{so}(2^{\frac{n}{2}})).\end{aligned}$$

(3) If $n \equiv 3 \pmod{8}$, then

$$\begin{aligned} \dim \mathfrak{m} + 1 &\geq \delta_2(n) + \delta_3(n) = 2^{n-2} + 2^{\frac{n-3}{2}} + 2^{n-2} - 2^{\frac{n-3}{2}} = 2^{n-1} \\ &= \dim_{\mathbb{R}}(\mathfrak{su}(2^{\frac{n-1}{2}})) + 1. \end{aligned}$$

(4) If $n \equiv 4 \pmod{8}$, then

$$\begin{aligned} \dim \mathfrak{m} &\geq \delta_2(n) + \delta_3(n) = 2^{n-2} + 2^{\frac{n}{2}-1} + 2^{n-2} = 2^{\frac{n-2}{2}}(2^{\frac{n}{2}} + 1) \\ &= \dim_{\mathbb{R}}(\mathfrak{sp}(2^{\frac{n-2}{2}})). \end{aligned}$$

(5) If $n \equiv 5 \pmod{8}$, then

$$\begin{aligned} \dim \mathfrak{m} &\geq \delta_2(n) + \delta_3(n) = 2 \left(2^{n-2} + 2^{\frac{n-3}{2}} \right) = 2^{\frac{n-1}{2}}(2^{\frac{n-1}{2}} + 1) \\ &= \dim_{\mathbb{R}}(\mathfrak{sp}(2^{\frac{n-3}{2}}) \oplus \mathfrak{sp}(2^{\frac{n-3}{2}})). \end{aligned}$$

(6) If $n \equiv 6 \pmod{8}$, then

$$\begin{aligned} \dim \mathfrak{m} &\geq \delta_2(n) + \delta_3(n) = 2^{n-2} + 2^{n-2} + 2^{\frac{n}{2}-1} = 2^{\frac{n-2}{2}}(2^{\frac{n}{2}} + 1) \\ &= \dim_{\mathbb{R}}(\mathfrak{sp}(2^{\frac{n-2}{2}})). \end{aligned}$$

(7) If $n \equiv 7 \pmod{8}$, then

$$\begin{aligned} \dim \mathfrak{m} + 1 &\geq \delta_2(n) + \delta_3(n) = 2^{n-2} - 2^{\frac{n-3}{2}} + 2^{n-2} + 2^{\frac{n-3}{2}} = 2^{n-1} \\ &= \dim_{\mathbb{R}}(\mathfrak{su}(2^{\frac{n-1}{2}})) + 1. \end{aligned}$$

5. GENERALIZED SPIN REPRESENTATIONS OF THE SPLIT REAL E_n SERIES AND THE RESULTING QUOTIENTS

The example of a generalized spin representation of the maximal compact subalgebra of the split real Kac–Moody Lie algebra of type E_{10} described in [DKN06], [DBHP06], [HKL13] generalizes directly to the whole E_n series as follows.

Let $n \in \mathbb{N}$, let \mathfrak{g} be the split real Kac–Moody Lie algebra of type E_n , let \mathfrak{k} be its maximal compact subalgebra, and let X_i , $1 \leq i \leq n$, be the Berman generators of \mathfrak{k} (cf. [Ber89, Theorem 1.31], [HKL13, Theorem 2.1]) enumerated in Bourbaki style as shown in Figure 1, i.e., $X_1, X_3, X_4, \dots, X_n$ belong to the A_{n-1} subdiagram, generating $\mathfrak{so}(n)$, and X_2 to the additional node. As in Section 2 let q be the standard positive definite quadratic form on \mathbb{R}^n and let $C = C(\mathbb{R}^n, q)$ be the corresponding Clifford algebra, considered as a Lie algebra.

Proposition 5.1. *Let $n \geq 3$. The assignment*

- $X_1 \mapsto v_1 v_2$,
- $X_2 \mapsto v_1 v_2 v_3$,
- $X_j \mapsto v_{j-1} v_j$ for $3 \leq j \leq n$

*defines a Lie algebra homomorphism ρ from \mathfrak{k} to the Lie subalgebra \mathfrak{m} of C generated by $v_1 v_2 v_3$ and by $v_i v_{i+1}$, $1 \leq i < n$, called the **standard generalized spin representation of \mathfrak{k}** .*

Proof. The proof is based on the criterion established in [HKL13, Remark 4.5] and is exactly the same as in the E_{10} case discussed in [HKL13, Example 4.1]. \square

Proof of Theorem A. By [HKL13, Theorem 4.11] and since E_n is simply laced and connected for $n \geq 4$, the image \mathfrak{m} of ρ is semisimple and compact. By Lemma 3.2 and Consequence 4.2, $\dim_{\mathbb{R}}(\mathfrak{m})$ is at least as large as the dimension of the semisimple maximal compact Lie subalgebra of C as given in Corollary 2.2. The claim follows. \square

Proof of Theorem B. Let \mathfrak{g} be a semisimple split real Lie algebra of type $E_4 = A_4$, $E_5 = D_5$, E_6 , E_7 or E_8 and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ its Iwasawa decomposition. Since $\dim_{\mathbb{R}}(\mathfrak{k}) = \dim_{\mathbb{R}}(\mathfrak{n})$, from the combinatorics of the respective root system we conclude that the maximal compact Lie subalgebra \mathfrak{k} has dimension

$$\begin{aligned} 10 &= \frac{4 \cdot 5}{2} = \frac{2^{\frac{4}{2}} \cdot (2^{\frac{4}{2}} + 1)}{2} = \dim_{\mathbb{R}}(\mathfrak{sp}(2)) = \dim_{\mathbb{R}}(\mathfrak{so}(5)) && \text{if } n = 4, \\ 20 &= 2 \cdot 10 = \dim_{\mathbb{R}}(\mathfrak{sp}(2) \oplus \mathfrak{sp}(2)) = \dim_{\mathbb{R}}(\mathfrak{so}(5) \oplus \mathfrak{so}(5)) && \text{if } n = 5, \\ 36 &= 4 \cdot 9 = 2^{\frac{6-2}{2}} (2^{\frac{6}{2}} + 1) = \dim_{\mathbb{R}}(\mathfrak{sp}(2^{\frac{n-2}{2}})) && \text{if } n = 6, \\ 63 &= 2^6 - 1 = \dim_{\mathbb{R}}(\mathfrak{su}(8)) && \text{if } n = 7, \\ 120 &= \frac{16 \cdot 15}{2} = \frac{2^{\frac{8}{2}} \cdot (2^{\frac{8}{2}} - 1)}{2} = \dim_{\mathbb{R}}(\mathfrak{so}(16)) && \text{if } n = 8. \end{aligned}$$

For $n \geq 4$ we may now apply Theorem A and deduce that the standard generalized spin representation ρ has to be injective in these cases.

This leaves the case $E_3 = A_2 \oplus A_1$. Since this diagram is not irreducible, [HKL13, Theorem 4.11] only implies that $\text{im}(\rho) = \mathfrak{m}$ is compact but not that it is semisimple (and indeed, it is not). However, $n = 3$ is also an exceptional case for Lemma 3.2. Taking that into consideration, it follows that $\dim_{\mathbb{R}}(\mathfrak{m}) \geq 2^2$ ($1, v_1 v_2, v_2 v_3, v_1 v_2 v_3$ is a basis of \mathfrak{m}). On the other hand, the Clifford algebra C is isomorphic to $M(2, \mathbb{C})$, hence $\mathfrak{k} \cong \mathfrak{u}(2)$, and this has dimension 4. Thus ρ is also injective when $n = 3$. The claim follows. \square

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JLU GIESSEN, MATHEMATISCHES INSTITUT, ARNDTSTRASSE 2, 35392 GIESSEN, GERMANY
E-mail address: max.horn@math.uni-giessen.de

JLU GIESSEN, MATHEMATISCHES INSTITUT, ARNDTSTRASSE 2, 35392 GIESSEN, GERMANY
E-mail address: ralf.koehl@math.uni-giessen.de